RF-3608/200

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M. A. / M. Sc. (Part - II) Examination

April / May – 2010

Mathematics : Paper - 502

(Advanced Abstract Algebra)

Time : 3 Hours] [Total Marks : **70**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી. **Fillup strictly the details of ☛ signs on your answer book.**	Seat No. :
Name of the Examination : ☛ **M. A. / M. Sc. - 2**	
Name of the Subject : ☛ **MATHEMATICS - 502**	
☛ Subject Code No. : **3 6 0 8** ☛ Section No. (1, 2,.....) : **NIL**	Student's Signature

(2) Each question carry equal marks.
(3) Attempt all questions.
(4) All questions are compulsory.

Q1(a) Define a Composition series of a Group. Prove that every finite group has a composition series.

(b) Define a constructible number. Can you trisect an angle of 60 degree with the compass and a straightedge only? Prove.

(c) Define a Simple Group. Prove that a group having 72 elements cannot be simple.

OR

Q1(a) Define an internal direct product of normal groups. Suppose, that G is an internal direct product of $N_1, N_2, \ldots, N_n$. Then for i≠j, $N_i \cap N_j = \{e\}$ and if a∈ N_i, b∈ N_j then a.b=b.a. Prove

(b) Define a p-sylow subgroup. State and prove the first part of Sylow theorem.

(c) Define an order of an element in a Group. If p is prime divisor of an order of a group G,then prove that G has an element of order p.

Q2(a) Let G be a nilpotent Group. Prove that every subgroup and every homomorphic image of G are nilpotent.

(b) Prove that a finite Group is solvable if and only if its composition factors are cyclic groups of prime orders.

(c) For a field Q of rational numbers. Prove that, $Q(\sqrt{2}, \sqrt{3}) = Q(\sqrt{2} + \sqrt{3})$.

OR

Q2(a) Define the splitting field of a polynomial. Prove that any two splitting fields of a polynomial over the same field are isomorphic.

(b) If p(x) is an irreducible polynomial in F[x] and v is a root of p(x) then F(v) is isomorphic to F (w) where w is a root of $p'(t)$. Moreover, this isomorphic σ can be so choosen that

(i) $v\sigma = w$ (ii) $\propto \sigma = \propto$ for $\forall \propto \in F$.

(c) Define a fixed field. Prove that fixed field of a group G is a subfield of K where F is contained in K.

Q3(a) If K and L are algebraic closures of a field F. Prove that K≅L under the isomorphism, which fixes every element of F.

(b) If K is a finite extension of a field f. Then prove that $G(K:F)$ is a finite Group and $O(G(K:F)) \leq [K:F]$.

(c) Define a simple extension of a field. Prove that a finite separable extension of a field is a simple extension.

OR

Q3(a) Define a Galois Group. State and Prove the fundamental theorem of Galois Group.

(b) Define algebraic closure of a field. Prove that every field has an algebraic closure.

(c) Let E be a finite separable extension of a field F. Show that E is a normal extension of F if and only if F is the fixed field of G(E : F).

Q4(a) Prove that every submodule and every homomorphic image of a noetherian module is again noetherian.

(b) Define a free module. Prove that each finitely generated module is a homomorphic image of a finitely generated free module.

(c) Let M be a Simple R-module. Then prove that $Hom_R(M, M)$ is a division ring.

OR

Q4(a) Define a Submodule. Let $(N_i)_{i\in\Lambda}$ be a family of R-Submodules of an R-module M and then prove that $\cap N_i$ is also an R-Submodule.

(b) Let R is a noetherian ring. Then prove that a.b=1 for a ,b ϵ R if and only if b. a = 1 .

(c) Define a Linearly independent Subset of an R-module M. Let M be a finitely generated free module over a commutative ring R. Then, prove that all the basis of M are finite.

Q5(a) If A and B are R-submodules of an R-module M. Show that, $\frac{A+B}{A} \cong \frac{B}{A\cap B}$ as R-modules.

(b) Define elementary row operations on an n x n matrix. Let A be an m x n matrix over R. Then prove the following:

(i) $E_{ij} = 1 - e_{ii} - e_{jj} + e_{ij} + e_{ji}$ *,then* $E_{ij}A(AE_{ij})$ is the matrix obtained from A by interchanging the i^{th} and j^{th} rows. Also, $E_{ij}^{-1} = E_{ij}$.

(ii) *If* $L_i(\propto) = 1 + (\propto - 1)e_{ii}$, and $\propto$ is an invertible element in R then $L_i(\propto)A[AL_i(\propto)]$ is the matrix obtained from A by multiplying the i^{th}row by $\propto$. also$L_i^{-1}(\propto) = 1 + (\propto^{-1} - 1)e_{ii}$

(iii)*If* $M_{ij}(\propto) = 1 + \propto e_{ij}$, then $M_{ij}(\propto)A[AM_{ji}(\propto)]$ is the matrix obtained from A by multiplying the j^{th} row by $\propto$ and adding it to the i^{th}row. Also, $M_{ij}^{-1}(\propto) = 1 - \propto e_{ij}$, $(i \neq j)$.

(c) If M is an R-module and $x \in M$, then show that the set $K = \{rx + nx / r \in R, n \in Z\}$ is their smallest R-submodule containing x. Also, prove that $K \cong Rx$ if R has unity.

OR

Q5(a) Let $(N_i)_{i\in\Lambda}$ be a family of R-modules of an R-module M. Then following are equivalent.

(i) $\sum_{i\in\Lambda} N_i$ is a direct sum.

(ii) $0 = \sum_i x_i \epsilon \sum_{i\in\Lambda} N_i \Rightarrow x_i = 0 \ \forall i.$

(iii) $N_i \cap \sum_{\substack{j\in\Lambda \\ j\neq i}} N_j = (0)$, $i \in \Lambda$

(b) Let A and B be R-submodules of R-modules M and N respectively. Show that, $\frac{MXN}{AXB} = \frac{M}{A} X \frac{N}{B}$

(c) Define a free module. Prove that the submodules of the quotient module M/N are of the form U/N, where U is a submodule of M containing N. .